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CLASSIFICATION OF HOPF ALGEBRAS OF DIMENSION 18

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ABSTRACT

This paper contributes to the classification problems of finite dimensional Hopf algebras H over an algebraically closed field $\mathbf{k}$ of characteristic zero. It is shown that for a non-semisimple Hopf algebra H of dimension 18 either H or H^* is pointed.

0. Introduction.

The classification of finite dimensional Hopf algebras has been developed rapidly since the end of 90's. D. Ştefan classified Hopf algebras in dimensions less than 12 [11]. It is shown by S.-H. Ng that a Hopf algebra of dimension p^2 is isomorphic to a group algebra, the dual of a group algebra or a Taft algebra [8]. For the pq dimensional Hopf algebras H where p, q are distinct primes, classification is still open in general. However many results have been obtained. N. Andruskiewitsch and S. Natale proved that $15, 21$ or 35-dimensional H are semisimple [1]. Furthermore M. Beattie and S. Dăscălescu settled the dimensions 14,55,65,77,91 and 143 in [2]. Other recent results of pq-dimensional cases are as follows. If primes p, q are twine primes p, $p + 2$ [9] or $p = 2$ [10] then H is semisimple by Ng. P. Etingof and S. Gelaki proved [3] that if $q \leq 2p + 1$, then H is semisimple. For the case $q \geq 2p+1$, using some generalization of the method in [1] and [2], it is shown in [4] that there is no non-semisimple Hopf algebras of dimension 33,39,57,85,95,115,119,133,145,161,203,319 or 377.

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On the other hand, other than the cases of prime and pq dimensions, the classification is settled only in dimensions 8 [11] and 12 [7]. Most recently, G. A. Garcia discussed in dimensions p^3, and classified quasi-triangular Hopf algebras of dimension 27 [5].

In this paper, we apply the method in [4] to Hopf algebras of dimension 18. We show the following

Theorem 0.1: *If H is a non-semisimple Hopf algebra of dimension 18 over an algebraically closed field of characteristic zero, then either H or H^* is pointed.*

1. Preliminaries

Throughout this paper, H is a finite dimensional Hopf algebra over an algebraically closed field $\mathbf{k}$ of characteristic zero, and Δ, ϵ, S denote the comultiplication, the counit, the antipode, respectively.

The n-th term of the coradical filtration of H is $H_n = \bigwedge^{n+1} H_0$, where $H_0 = \bigoplus_i C_i$ is the coradical of H. As $\mathbf{k}$ is algebraically closed, there exists a coalgebra projection $\pi : H \to H_0$ and $H = H_0 \oplus I$, where $\ker \pi = I$(see [6, 5.4.2]). Setting $\rho_l = (\pi \otimes \mathrm{id})\Delta$ and $\rho_r = (\mathrm{id} \otimes \pi)\Delta$, H is a H_0-bicomodule with the structure maps ρ_l and ρ_r. H_0, H_n, I are H_0-subbicomodules of H. Any H_0-bicomodule is a direct sum of simple H_0-subbicomodules and a simple H_0-bicomodule has coefficient coalgebras $(C_i\ ,\ C_j)$ and its dimension is $\sqrt{(\dim C_i)(\dim C_j)}$.

Let $P_n\,, n = 1, 2, \ldots$ be defined inductively by:

$$P_1 = \{x \in H; \Delta(x) - \rho_l(x) - \rho_r(x) = 0\},$$
$$P_n = \left\{x \in H; \Delta(x) - \rho_l(x) - \rho_r(x) \in \sum_{1 \le i \le n-1} P_i \otimes P_{n-i}\right\}, \quad n \ge 2.$$

Then $P_n = H_n \cap I$ and P_n are H_0-subbicomodules of I, due to Nichols (see [1, Lemma 1.1]). We denote by $P_n^{C_i,\,C_j}$ the isotypic component of simple subbicomodule of P_n with coalgebra of coefficients $(C_i\ ,\ C_j)$. We say the subspace $P_n^{C_i,\,C_j}$ is non-degenerate if $P_n^{C_i,\,C_j} \not\subset P_{n-1}$.

The following result is from [2].

Proposition 1.1: *If there is no non-trivial skew primitives then there exists a simple subcoalgebra C $(\dim C \ge 4)$ of H such that $P_1^{1,\,C} \neq 0$.*

The next Lemmas were obtained in [4]. Lemma 1.2 is a generalization of [1, Corollary 1.3].

LEMMA 1.2: $\dim P_n^{C,\,D} = \dim P_n^{gC,\,gD} = \dim P_n^{Cg,\,Dg} = \dim P_n^{S(D),\,S(C)}$ *for* $g \in G(H)$.

LEMMA 1.3: *Suppose there exist simple subcoalgebras* C *and* D *such that* $P_m^{C,\,D}$ *is non-degenerate. Assume further* $\dim C \neq \dim D$ *or* $\dim P_m^{C,\,D} - \dim P_{m-1}^{C,\,D} \neq \dim C$. *Then there exists a simple subcoalgebra* E *such that* $P_l^{C,\,E}$ *is non-degenerate for some* $l \geq m+1$.

2. Proof of Theorem 0.1.

Throughout this section, H be a non-semisimple and non-pointed Hopf algebra of dimension 18 over **k**. First we show the following

LEMMA 2.1: *Let* H *be a Hopf algebra as above and* $|G(H)| > 1$. *Then* H *contains a Taft Hopf algebra* $T(3)$ *of dimension* 9 *and* $|G(H)| = 3$.

Proof. First we suppose that H has no non-trivial skew primitive element. Hence, by Proposition 1.1, there exists a simple subcoalgebra C with $\dim C \geq 4$ such that $P_1^{1,\,C} \neq 0$. Thus $P_1^{S(C),\,1} \neq 0$ by Lemma 1.2. It follows from Lemma 1.3 that there exist a grouplike element h and simple subcoalgebra E with $\dim E = \dim C$ such that $P_n^{1,\,h}$ and $P_m^{S(C),\,E}$ are non-degenerate for some integers $m, n \geq 2$. By Lemma 1.2, $\dim P_1^{g,\,gC} = \dim P_1^{gS(C),\,g} = \dim P_1^{1,\,C}$ for all $g \in G(H)$. Note that $\{P_1^{g,\,gC} : g \in G(H)\} \cup \{P_1^{gS(C),\,g} : g \in G(H)\}$ is a set of linearly independent subspaces of H. Therefore,

$$\begin{aligned} 18 = \dim H &\geq \dim \left(H_0 + P_m^{S(C),\,E} + \sum_{g \in G(H)} P_1^{g,\,gC} + P_1^{gS(C),\,g} + P_n^{g,\,gh} \right) \\ &\geq 2(|G(H)| + \dim C) + 2|G(H)| \dim P_1^{1,\,C} \\ &\geq 2(|G(H)| + \dim C + |G(H)|\sqrt{dimC}). \end{aligned}$$

This implies that $(|G(H)|, \dim C) = (1, 4)$ which contradicts the assumption $|G(H)| > 1$. Therefore, H has a non-trivial $(1, g)$-primitive element x for some $g \in G(H)$. Let L be the Hopf algebra generated by x, g. Then L is non-semisimple and pointed and so L is not isomorphic to H. By [1, Proposition 1.8], $\dim L$ has a square factor. Therefore, $\dim L = 9$ and so $L \cong T(3)$ by [11] or [9].

By the result above, $|G(H)| = 3k$ for some integer k. Since H is non-pointed, a simple subcoalgebra C with $\dim C \geq 4$ is contained in H_0. By counting dimensions, $\dim C = 4$ or 9. If $\dim C = 4$ then $3k | \dim H_{0,2}$ where $H_{0,2}$ is the sum of all 4-dimensional simple subcoalgebras of H [1, Lemma 2.1(i)]. And so $\dim H_0 \geq |G(H)| + \dim H_{0,2} \geq 15$. This contradicts $T(3) \subset H$. Thus $\dim C = 9$ hence $H \cong T(3) \oplus C$. ∎

Remainder of the proof of Theorem 0.1. We assume further H^* is non-pointed. By [10, Corollary 2.2], $G(H)$ or $G(H^*)$ is not trivial. By duality, we may assume that $|G(H^*)| > 1$. By Lemma 2.1, $T(3) \subset H^*$ and so there exists a Hopf algebra projection $\Pi : H \to T(3)^* \simeq T(3)$. Let $H^{co\Pi}$ be the coinvariant $\{x \in H : (\mathrm{id} \otimes \Pi)\Delta(x) = x \otimes \Pi(1)\}$. Then $H^{co\Pi}$ is a left coideal subalgebra of H, $\dim H^{co\Pi} = 2$.

If $\dim Soc(H^{co\Pi}) = 2$ then $H^{co\Pi}$ is a subHopf algebra of H and $H^{co\Pi} \simeq \mathbf{kC_2}$. This contradicts Lemma 2.1 which implies that $|G(H)| = 3$. Thus $Soc(H^{co\Pi}) = \mathbf{k}1$. The $H^{co\Pi}$ is expressed as $\mathbf{k}1 \oplus \mathbf{k}x$ where $x \in (H^{co\Pi})^+$. Since $H^{co\Pi}$ is a left coideal, $\Delta(x)$ can be expressed as $\alpha \otimes 1 + \beta \otimes x$. In this case, the α above is equal to x and the β above is a non-trivial grouplike element by calculating $\Delta^{(2)}(x)$ with the coassociativity of Δ. So x is a skew primitive element. By Lemma 2.1, the order of β is 3.

If x is a non-trivial skew primitive element, then Hopf algebra L generated by $\{x, \beta\}$ is non-semisimple and pointed. Thus $\dim L$ has a square factor and is a proper factor of 18. This implies that $\dim L = 9$ and hence $L \cong T(3)$. Therefore, $1, x, x^2$ are linearly independent which contradicts that $\dim H^{co\Pi} = 2$.

If x is a trivial skew primitive element, i.e. $x = k(1-\beta)$ for some $k \in \mathbf{k}^\times$, then the algebra generated by $\{1, x\}$ is the group algebra $\mathbf{k}[\beta]$ which is of dimension 3. This also contradicts that $\dim H^{co\Pi} = 2$.

This completes the proof of Theorem 0.1. ∎

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